國立中正大學
110學年度碩士班招生考試
試題
[第2節]

科目名稱	數學
系所組別	資訊工程學系 甲組 - 乙組

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始40分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

1. (12 points) For the matrix A and its reduced row echelon form are given below:

$$A = \begin{bmatrix} 5 & 15 & 5 & 0 & 4 \\ 4 & 12 & 4 & 5 & -3 \\ -2 & -6 & -2 & 0 & -2 \\ -2 & -6 & -2 & 1 & -5 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the following questions:

(a) (3 points) Find a basis for the null space of A.

(b) (3 points) Find a basis for the row space of A.

(c) (3 points) Find a basis for the column space of A.

(d) (3 points) Find the rank and the nullity of A.

2. (8 points) The following vectors

$$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \overrightarrow{v_3} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \overrightarrow{v_4} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

span a subspace V *of* R^3, but not a basis for V. Answer the following questions.

(a) (4 points) Choose a subset of $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ which forms a basis for V.

(b) (4 points) Extend this basis to a basis for R^3.

3. (10 points)Let $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ and let P be the plane through the origin spanned by $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

(a) (5 points) Find an orthonormal basis of P.

(b) (5 points) Find the point on P which is closest to the point (1,0,0).

4. (10 points) Let $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ denote the following vectors in R^3.

$$\overrightarrow{v_1} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \quad \overrightarrow{v_2} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix}$$

(a) (3 points) Find a vector $\overrightarrow{v_3}$ so that $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, $\overrightarrow{v_3}$ form an orthonormal basis B of R^3. How many choices are there for the answer?

(b) (3 points) Let T: $R^3 \rightarrow R^3$ denote the linear transformation that interchanges $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$ and has $\overrightarrow{v_2}$ as an eigenvector with eigenvalue -5. Write down $[T]_B$, the matrix of T with respect to the basis B.

(c) (4 points) Write down a product of matrices that equals the standard matrix of T.

5.(10 points) Briefly explain each of the following matrix factorization methods. You also need to specify the existing constraints for each matrix factorization.

(a) (5 points) QR decomposition

(b) (5 points) Singular value decomposition

6. (10 points) Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.
   (a) (2 points) $\forall x \exists y (x^2 = y)$
   (b) (2 points) $\forall x \exists y (x = y^2)$
   (c) (2 points) $\forall x (x^2 \neq x)$
   (d) (2 points) $\forall x (|x| > 0)$
   (e) (2 points) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$

7. (10 points) If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m.
(a) (5 points) Find an inverse of 72 modulo 233.
(b) (5 points) Solve the congruence $72\,x \equiv 6 \pmod{233}$

8. (10 points) How many numbers must be selected from the first 10 positive integers to guarantee that at least three pairs of these numbers add up to 11?

9. (10 points) A string that contains only 0s and 1s is called a binary string.
(a) (5 points) Find a recurrence relation for the number of binary strings of length $\boldsymbol{n}$ that do not contain two consecutive 0s.
(b) (2 points) What are the initial conditions?
(c) (3 points) How many binary strings of length 7 do not contain two consecutive 0s?

10. (10 points) The complementary graph $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.
(a) (5 points) If G is a simple graph with 20 edges and $\overline{G}$ has 16 edges, how many vertices does G have?
(b) (5 points) If the simple graph G has x vertices and y edges, how many edges does $\overline{G}$ have?